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Financial Risk Management
an application to MATLAB

University Paris-Dauphine
Master Degree & Pre-Doctoral Program

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Chapter 2: Systematic Risk, Asset Pricing in Intertemporal Models

Content of the Lecture

Objectives of the lecture

- Understanding the standard Lucas asset pricing model;
- Highlighting both the equity premium and riskless rate puzzles;
- Fixing these puzzles through adjustment costs or recursive preferences;

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1. Asset pricing in a pure exchange economy

Lucas Jr (1978) provides a theoretical model to examine the stochastic behaviour of equilibrium asset prices using a one-good, pure exchange economy populated by identical consumers. The analysis conducted by Lucas is done in a rational expectation equilibrium as defined by Muth (1961), this equilibrium allows assets prices to "fully reflect all available information" (Malkiel and Fama (1970)).

In this lecture, we employ perturbation methods of Juillard (2011) to perform higher order approximations to policy functions of models.

1.1 The model

Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. He wishes to maximize its welfare defined as its expected stream of discounted utilities:

$$\max_{\{c_t, z_{jt}, b_t\}} \mathbb{E}_t \left\{ \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{U}\left(c_{t+\tau}\right) \right\}, \tag{1}$$

where c_t is a stochastic process representing consumption of a single good, β is the discount factor, $\mathcal{U}(\cdot)$ is the current utility function and $\mathbb{E}_t\{\cdot\}$ is an expectations operator.

The consumption good is produced on n distinct productive units. Let y_{it} be the output of unit i in period t, $i = 1, ..., n$, and let $y_t = (y_{1t}, ..., y_{nt})$ be the output vector in t. Output is perishable, so that feasible consumption levels are those which satisfy:

$$0 \leq c_t \leq \textstyle\sum_{i=1}^{n} y_{it}. \tag{2}$$

Production is entirely "exogenous": no resources are utilized, and there is no possibility of affecting the output of any unit at any time. The motion of y_t is determined by a short term z_t and a long term A_t components:

$$y_t = z_t A_t \tag{3}$$
$$\text{with } \log(z_t) = \rho \log(z_{t-1}) + \sigma e_t, \tag{4}$$
$$\text{and } A_t = g A_{t-1}. \tag{5}$$

where z_t denotes a productivity AR(1) shock process with AR term $\rho \in [0, 1)$ and $e_t \sim \mathcal{N}(0, 1)$ is a normal innovation with standard deviation $\sigma \geq 0$; variable A_t is the long run component which can be associated to labour augmenting productivity with trend slope $g \geq 1$. When $g = 1$, the economy is stationary in the long run (inconsistent with the data) while when $g > 1$ the economy is steadily increasing at a constant rate (0.5% for instance for the US economy on a quarterly basis). Here, z_t generates short run fluctuations around the trend g.

Ownership in these productive units is determined each period in a competitive stock market. Each unit has outstanding one perfectly divisible equity share. A share entitles its owner as of the beginning of t to all of the unit's output in period t. Shares are traded, after payment of real dividends d_{it}, at a competitively determined price vector $p_t = (p_{1t}, ..., p_{nt})$. Let $z_t = (z_{1t}, ..., z_{nt})$ denote a consurner's beginning-of-period share holdings. Finally, to create a portfolio choice problem, the household has the possibility to buy riskless assets b_t with real interest rate r_t. This riskless asset allows the representative household to postpone consumption without facing uncertainty as the riskless interest rate r_{t-1} is known in $t-1$. The budget constraint of the representative household reads as:

$$c_t + \sum_{i=1}^{n} p_{it} z_{it} + b_t \leq \sum_{i=1}^{n} \left[d_{it} + p_{it}\right] z_{it-1} + \left(1 + r_{t-1}\right) b_{t-1} \tag{6}$$

The first order condition of the problem in (1) under the budget constraint in **??** are given by:

$$p_{it} = \beta \mathbb{E}_t \left\{ \frac{\mathcal{U}'\left(c_{t+1}\right)}{\mathcal{U}'\left(c_t\right)} \left[d_{it+1} + p_{it+1} \right] \right\}, \, j = 1, .., n \tag{7}$$

$$1 = \beta \mathbb{E}_t \left\{ \frac{\mathcal{U}'\left(c_{t+1}\right)}{\mathcal{U}'\left(c_t\right)} \right\} \left(1 + r_t\right) \tag{8}$$

1.2 Assets rates of return

Recall that the optimal portfolio allocation between risky and safe assets is given by the optimal first order conditions of the representative household utility maximization problem:

$$\mathcal{U}'\left(c_t\right) p_{it} = \beta \mathbb{E}_t \left\{ \mathcal{U}'\left(c_{t+1}\right) \left[d_{it+1} + p_{it+1} \right] \right\} \tag{9}$$

$$M_{t,t+1} = \beta \mathbb{E}_t \left\{ \frac{\mathcal{U}'\left(c_{t+1}\right)}{\mathcal{U}'\left(c_t\right)} \right\} \tag{10}$$

where $M_{t,t+1}$ is the marginal rate of substitution of consumption. In the first equation, the investor willing to buy in t an additional asset z_i must sacrify p_{it} units of consumption now, leading to a total utility loss of: $\mathcal{U}'\left(c_t\right) p_{it}$. This investment generates in terms of assets and dividends in t+1: $d_{it+1} + p_{it+1}$, the consumption of this amount will enhance the utility of $\mathcal{U}'\left(c_{t+1}\right)\left[d_{it+1} + p_{it+1}\right]$. The intertemporal no-arbitrage condition determining the asset allocation is thus given by:

$$\underbrace{\mathcal{U}'\left(c_t\right) p_{it}}_{\text{utility loss in t}} = \underbrace{\beta \mathbb{E}_t \left\{ \mathcal{U}'\left(c_{t+1}\right) \left[y_{it+1} + p_{it+1} \right] \right\}}_{\text{discounted expected utility gain in t+1}},$$

which states that the optimal portfolio is obtained when the marginal cost of investing in asset i in t equals its expected marginal return in $t+1$.

Similarly, buying an additional unit b_t provides a utility loss of $\mathcal{U}'\left(c_t\right)$ but offers an expected utility gain of $\beta \mathbb{E}_t \left\{\mathcal{U}'\left(c_{t+1}\right)\right\}\left(1 + r_t\right)$ in $t+1$. Optimal allocation of safe assets thus implies that the investor must buy safe assets until the marginal cost of investing in t equals its expected marginal benefit in $t+1$.

$$\underbrace{\mathcal{U}'\left(c_t\right)}_{\text{utility loss in t}} = \underbrace{\beta \mathbb{E}_t \left\{\mathcal{U}'\left(c_{t+1}\right)\left(1 + r_t\right)\right\}}_{\text{discounted expected utility gain in t+1}}$$

From these two equations, it is possible to derive the return from holding one unit of safe and risky asset between t and t+1:

$$1 + r^k_{it+1} = \beta \mathbb{E}_t \left\{ \frac{d_{it+1} + p_{it+1}}{p_{it}} \right\},$$
$$1 + r_t = \frac{1}{\mathbb{E}_t \left\{ M_{t,t+1} \right\}},$$

here, a key difference between r^k_{t+1} and r_t lies in the fact that r^k_{t+1} is observed *ex post* (once the investment decision is taken) while r_t is known *ex ante* before buying safe assets. By so, there is uncertainty for the real return of safe assets contrary to r^k_{t+1} which involves uncertainty regarding expected dividends and future asset prices. Using r^k_{t+1} and r_t allows to rewrite asset price in Equation 7 and Equation 8 and illustrate the uncertainty regarding the return of risky assets:

$$1 = \mathbb{E}_t \left\{ M_{t,t+1} \left(1 + r^k_{t+1} \right) \right\} \tag{11}$$
$$1 = \mathbb{E}_t \left\{ M_{t,t+1} \right\} \left(1 + r_t \right) \tag{12}$$

In these equations $M_{t,t+1}$ is the pricing kernel, it is also referred as the stochastic discount factor or the intertemporal marginal rate of substitution of consumption.

1.3 Closing the model

To close the model, the production of the economy is completely consumed:

$$c_t = y_t = \sum\nolimits_{i=1}^{n} y_{it}, \tag{13}$$

so that all the endowment of the economy is consumed each period.

In addition, dividends are measures by the total production:

$$d_{it} = y_{it}.$$

The portfolio is normalized to one with $n = 1$:

$$\sum\nolimits_{i=1}^{n} z_{it} = 1, \tag{14}$$

implying that the representative holds all the assets available in the economy. In addition, z_{it} can be interpreted as the share of asset i in household's portfolio.

His level of debt is zero:

$$b_t = 0 \tag{15}$$

which is consistent with a closed economy setup where households cannot use external borrowing to finance current account deficits. Implicitly, borrowing/saving is only allowed between households.

The preferences are the crucial ingredient of the analysis. The usual utility is given by a CRRA type function:

$$U\left(c_t\right) = \frac{c_t^{1-\gamma}}{1-\gamma} \tag{16}$$

where $\gamma > 0$ is the well-known Arrow-Pratt coefficient for risk aversion. This parameter features risk neutrality for $\gamma = 0$, while when $\gamma \to 1$, the utility function becomes logarithmic in consumption.

This utility has an important limitation because of its isoleastic functional form. In particular, the parameter γ determines at the same time the risk aversion coefficient (γ) and the intertemporal elasticity of substitution ($1/\gamma$). Risk aversion coefficient has implications on the intertemporal allocation of consumption for different possible states of the nature (*i.e.* under uncertainty) while intertemporal elasticity of substitution only affects the intertemporal marginal rate of consumption (without uncertainty).[3]

Finally, our model is not stationary because of the long trend on labor productivity $A_t = gA_{t-1}$. The model needs to be stationarized before local approximation. Let's define stationarized variables as follows:

$$\tilde{c}_t = C_t/A_t, \;\; \tilde{\lambda}_t = U'\left(c_t\right)/A_t^{-\gamma}$$

Under this definition, our model reads as a set:

$$\begin{aligned}
\tilde{\lambda}_t &= \hat{c}_t^{-\gamma}, \\
p_t &= \tilde{\beta}\mathbb{E}_t\left\{\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\left[z_{t+1}+p_{t+1}\right]\right\} \\
\left(1+r_t\right)\tilde{\beta}\mathbb{E}_t\left\{\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\right\} &= 1, \\
\left(1+\mathbb{E}_t r_{t+1}^k\right) &= \tilde{\beta}\mathbb{E}_t\left\{\frac{z_{t+1}+p_{t+1}}{p_t}\right\}, \\
\log(z_t) &= \rho\log(z_{t-1})+\sigma e_t,
\end{aligned}$$

where $\tilde{\beta} = g^{-\gamma}\beta$ as the growth rate of the economy affects the discount factor.

1.4 Calibration

Here we don't use the calibration of Lucas as it fails at reproducing standard business cycles second moments statistics. In particular, we calibrate the productivity shock (i.e. $\rho = 0.8$ and $\sigma = 0.055$) in order to reproduce the annual standard deviation of the growth rate of real consumption. We fix $\beta = 0.99$ to generate an annual riskless rate of 4%, while $\gamma = 2$ in order to have enough uncertainty up to second order. Finally, real per capita consumption has been steadily growing at an annual rate of 2%, thus we fix $g = 1.02$ in order to reproduce a similar trend as observed for the US economy.

1.5 A discussion on the risk

1.5.1 The equity premium

Using preferences in Equation 16, we can rewrite Equation 11 and apply logs:

$$1 = \mathbb{E}_t\left\{M_{t,t+1}\left(1+r_{t+1}^k\right)\right\} \tag{17}$$

$$0 = \mathbb{E}_t\ln(M_{t,t+1})+\mathbb{E}_t r_{t+1}^k \tag{18}$$

It is important to stress that up to a second order approximation of equilibrium conditions of the model, the unconditional mean of each endogenous variable is determined

[3] A simple way to disentangle these two coefficients is to introduce recursive preferences: $\mathcal{U}_t = [(1-\beta)\,c_t^{1-\rho} + \beta[\mathbb{E}_t\mathcal{U}_{t+1}^{1-\gamma}]^{\frac{1-\rho}{1-\gamma}}]^{\frac{1}{1-\rho}}$.

by its steady state + a fraction of the variance: $\ln E[X_t] = E[\ln X_t] + \frac{1}{2}var(\ln X_t)$, where $E[\ln X_t]$ is the first order term of Taylor series (or/and the steady state) and $\frac{1}{2}var(\ln X_t)$ the second order term.[4] For variables products, the unconditional mean up to second order is given by:[5]

$$0 = E[\ln M_{t,t+1}] + E\left[r^k_{t+1}\right] + \frac{1}{2}var(\ln M_{t,t+1}) + \frac{1}{2}var\left(r^k_{t+1}\right) + cov(\ln M_{t,t+1}, r^k_{t+1}).$$

Recalling that, $E[r_t] = -E[\ln M_{t,t+1}] - \frac{1}{2}var(\ln M_{t,t+1})$,[6] we can simplify injecting $E[r_t]$:

$$0 = E\left[r^k_{t+1}\right] - E[r_t] + \frac{1}{2}var\left(r^k_{t+1}\right) + cov(\ln M_{t,t+1}, r^k_{t+1}). \tag{19}$$

Finally, the equity premium reads as:

$$EP_t = \frac{\left(1 + r^k_{t+1}\right)}{(1 + r_t)} \tag{20}$$

Taken in logs, the unconditional mean is given by:

$$E[\ln EP_t] = E\left[r^k_{t+1}\right] - E[r_t] + \frac{1}{2}var\left(r^k_{t+1}\right) \tag{21}$$

Using Equation 19, the analytical value of the equity premium emerges:

$$E[\ln EP_t] = -cov(\ln M_{t,t+1}, r^k_{t+1}) \tag{22}$$

This equation illustrates the risk premia in simple intertemporal models. The equity premium is endogenously determined by the negative of the unconditional covariance of the stochastic discount factor with the asset returns. Consumers demand a high premium in order to hold this asset when the return on an asset co-varies negatively with the stochastic discount factor for a simple reason: the asset tends to have low returns and hence decrease the value of wealth precisely when consumers need it more (when the marginal utility of consumption is lower than in the future, i.e. when the stochastic discount factor is high: remember that $M_{t,t+1}$ is high when marginal utility of consumption today is low compared to the future one).

In addition, with $\ln M_{t,t+1} = \ln\beta - \gamma\mathbb{E}_t\{\ln\Delta\tilde{c}_{t+1}\}$, we can deduct that $cov(\ln M_{t,t+1}, X_t) = -\gamma cov(\ln\Delta\tilde{c}_{t+1}, X_t)$. Then:

$$E[\ln EP_t] = \gamma cov(\ln\Delta\tilde{c}_{t+1}, r^k_{t+1}), \tag{23}$$

this equations provides an alternative interpretation of the equity premium as it is defined by the product between the relative risk aversion coefficient γ and the covariance between expected consumption growth and asset returns. Intuitively, a large equity

[4]Using a first order approximation, we have $\ln E[X_t] = E[\ln X_t]$ while up to second order, $\ln E[X_t] \leq E[\ln X_t]$ because of the second order term. This is also refered to as Jensen's inequality. In addition, since X_t is also homoskedastic, conditional second moments $var_t(X_t)$ are equal to unconditional second moments $var(X_t)$, so I have already dropped the time subscript on second moments.

[5]Recall that for the asymptotic mean of an endogenous variable up to second order and taken in logs reads as follows: $\ln E[X_tY_t] = E[\ln X_t] + E[\ln Y_t] + \frac{1}{2}var(\ln X_t) + \frac{1}{2}var(\ln Y_t) + cov(X_t, Y_t)$.

[6]In this expression, there is no covariance between r_t and $M_{t,t+1}$ as the return for safe assets is known with certainty and uncorrelated with the stochastic discount factor. This is because r_t is known before investment decisions.

premium is required when (for a given risk aversion) there is a high covariance between returns and consumption, in this case the asset delivers low returns when consumption is low (when marginal utility of consumption is high). Put differently with $\ln \Delta c_{t+1}$ representing uncertainty regarding the future states of the economy, the investor requires an high equity premium to undertake the risk.

Exercise 1

In this exercise, we investigate why uncertainty (and in turn the equity premium) requires a second order approximation to the model equilibrium conditions. To do so, fill the following table by answering questions.

	steady state	order 1	order 2	order 3
Means				
$E\left(r_t^k\right)$ - risky rate	....	....	....	....
$E\left(r_t\right)$ - riskless rate	....	....	....	....
$E\left(EP_t\right)$ - equity premium	....	....	....	....
Ratios				
$\frac{E(EP_t)}{\sigma(r_t^k)}$ - sharpe ratio	....	....	....	....

1. Run lucas.mod file, in the command window you should be able to find "STEADY-STATE RESULTS:" to grab the value of the steady state of the model.

2. At the end of the file, you should find 'stoch_simul(...)' function taking as argument 'order=2'. Change this value to evaluate the model up to first order, second order and third order and fill the table with asymptotic means reported in the command window right after 'APROXIMATED THEORETICAL MOMENTS'.

3. Recall that the approximation of a variable reads as:

$$E\left[X_t\right] \simeq \underbrace{\bar{X}}_{\text{steady state}} + \underbrace{E\left[X_t - X\right]}_{\text{1st order}} + \underbrace{\frac{1}{2}E\left[\left(X_t - X\right)^2\right]}_{\text{2nd order}} + \underbrace{\frac{1}{3}E\left[\left(X_t - X\right)^3\right]}_{\text{3rd order}} + ...$$

   Why does the evaluation of the equity premium $E\left[EP_t\right]$ requires higher order approximations? Where does uncertainty only appears up to second order?

4. Interpret the third order term.

Exercise 2

The point of this exercise is to highlight the equity premium puzzle. To do so, we are going to fill the following table.

1. Use data.xlsx and fill the column "data" in Table 1 (page 17).
2. Use lucas.mod to simulate the Lucas' model and fill the column 'Lucas simple' in Table 1 (page 17). considering a second order approximation to the policy function.
3. What is consistent or inconsistent with empirical evidence?
4. Mehra and Prescott (1985) had pointed out that this model was not able to replicate the equity premium and called this the "equity premium puzzle". Using a loop, find the value of γ, here are some tips about making the loop:

```
% go the end of the dynare file
% some stuff to define
for i = % fill the rest of the line
        % fill here to update the new value of gamma
        stoch_simul(order=2,nograph,noprint) ep;
        mean_ep = oo_mean(1);
        % fill here to save the mean value of ep in a matrix
end
% fill here to plot the result
```

5. Empirical evidence reports that γ is roughly between 0 and 2. Compare with the value of γ.

1.5.2 The riskless rate

Under these preferences, we can rewrite Equation 12 as:

$$(1+r_t) = \frac{g^\gamma}{\beta}\mathbb{E}_t\left\{\frac{\tilde{c}_{t+1}}{\tilde{c}_t}\right\}^\gamma \tag{24}$$

The GDP shock is defined by an AR(1) process with a stochastic innovation following a normal distribution. Given this shock specification, each random variable taken in logs follows a lognormal distribution which allows us to characterize second moment statistics. Applying the Taylor series up to second order to Equation 24, we find:

$$E\left[r_t\right] = -\ln\beta + \gamma\ln(g) - \frac{\gamma^2}{2}var\left(\Delta\ln\tilde{c}_{t+1}\right) \tag{25}$$

This result shows that the real rate of return of safe assets has 3 important drivers: (i) $-\ln\beta$ the discount factor so that the riskless rate increases in the households degree of impatience;[7] (ii) $\gamma \ln(g)$ the growth of consumption implying that an higher real interest rate encourages households to smooth/postpone consumption by providing more liquidities on financial markets; (iii) $var\left(\Delta \ln \tilde{c}_{t+1}\right)$ the variance of the expected growth rate of (detrended) consumption, the latter is a second order term that can be interpreted as a Keynesian precautionary saving to shield against expected uncertainty. When the economy is near its deterministic trend (i.e. for a low value of $var\left(\Delta \ln \tilde{c}_{t+1}\right)$), the uncertainty regarding future states of the nature is low as well, so households have no rationales for making precautionary saving. In addition, the degree of risk aversion γ has important implications regarding the riskless interest rate and uncertainty as it increases by $0.5\gamma^2$ the sensitivity of the riskless rate to the variance of expected consumption.

Exercise 3

The point of this exercise is analyze the term premium puzzle.

1. Let's assume that empirically, the real return from holding riskless Tbills is $E\left[r_t\right]$ =1%. Does Lucas' model replicate this interest rate? Explain your result using Equation 25

2. Weil (1989) had pointed out that this model was not able to replicate the riskless interest rate which was called this the "riskless rate puzzle". Using a loop, find the value of γ, here are some tips about making the loop:

```
% go the end of the dynare file
% some stuff to define
for i = % fill the rest of the line
        % fill here to update the new value of gamma
        stoch_simul(order=2,nograph,noprint) r;
        mean_ep = oo_mean(1);
        % fill here to save the mean value of ep in a matrix
end
% fill here to plot the result
```

3. Empirical evidence reports that γ is roughly between 1 and 2. Compare with the value of γ you have computed. Do we need a high or a low value of γ? Why does this matter regarding the equity premium puzzle?

4. Assuming that $\gamma = 2$, find the value of β that fixes the puzzle using a loop. Do you find it consistent with micro-evidence?

[7] Patient households tends to have an higher discount factor than impatient households as highlighted by Kiyotaki and Moore (1997).

2. Asset pricing in a production economy

In this section, we leave the Lucas' pure exchange economy and extend the framework to account for production factors (i.e. endogenous production) which offers to investors the possibility to endogenously affect output and resources over business cycles. We develop here a simple model featuring both consumption habits and adjustment costs on capital. The model is strongly inspired by Jermann (1998) and Beaubrun-Diant and Tripier (2005), these contributions aims at fixing some anomalies in intertemporal asset pricing theory. The aim of this section is to fix the equity premium puzzle through adjustment cost on capital goods production.

Here, we disentangle investors and firms. Investors maximize consumption by facing a trade-off between risky and riskless assets. Firms determine the level of output and the resources employed in the production process of goods, generating dynamics for both the capital goods price and the wage rate.

2.1 Households

Representative investors maximize expected lifetime utility of consumption, subject to a sequential budget constraint:

$$\max_{\{c_t,b_t,z_{it}\}} \mathbb{E}_t \left\{ \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\left(c_{t+\tau} - hc_{t-1+\tau}\right)^{1-\gamma}}{1-\gamma} \right\}, \tag{26a}$$

$$s.t. \;\; c_t + b_t + \textstyle\sum_{i=1}^{n} p_{it} z_{it} = (1+r_{t-1}) b_{t-1} + \sum_{i=1}^{n} \left[d_{it} + p_{it}\right] z_{it-1} \tag{26b}$$

where $h \in [0,1)$ denotes the degree of consumption habits as in Constantinides (1990) and $\gamma > 0$ the risk aversion parameter, b_t riskless bonds and z_{it} risky assets (shares) which are priced at a market price of p_{it}. Shareholding provides uncertain dividends d_{it} while bonds offers a certain interest rate r_t.

The problem of the representative households determining its consumption, its investment positions on riskless bonds and risky assets are determined by:

$$\begin{aligned} L = \sum_{\tau=0}^{\infty} \beta^{\tau} \{ & \frac{\left(c_{t+\tau} - hc_{t-1+\tau}\right)^{1-\gamma}}{1-\gamma} \\ & - \lambda_{t+\tau} \left[c_{t+\tau} + b_{t+\tau} + \sum_{i=1}^{n} p_{it+\tau} z_{it+\tau} - (1+r_{t-1+\tau}) b_{t-1+\tau} - \sum_{i=1}^{n} \left[d_{it+\tau} + p_{it+\tau}\right] z_{it-1+\tau} \right] \} \end{aligned}$$

where λ_t^c is the lagrange multiplier on the budget constraint.

First order conditions are given by:

$$\begin{aligned} c_t &: \lambda_t = \left(c_t - hc_{t-1}\right)^{-\gamma} - \beta h \left(\mathbb{E}_t \left\{c_{t+1}\right\} - hc_t\right)^{-\gamma}, \\ b_t &: \lambda_t = \beta \lambda_{t+1}^c (1+r_t) \\ z_{it} &: \lambda_t p_{it} = \beta \lambda_{t+1} \left[d_{it+1} + p_{it+1}\right] \end{aligned}$$

Substituting Lagrange multipliers, we can employ the pricing kernel $M_{t,t+1} = \beta \lambda_{t+1} / \lambda_t$ determining the riskless interest rate r_t and the risky asset price p_{it}:

$$\mathbb{E}_t \left\{M_{t,t+1}\right\} (1+r_t) = 1, \tag{27}$$

$$\mathbb{E}_t \left\{M_{t,t+1} \left[d_{it+1} + p_{it+1}\right]\right\} = p_{it}. \tag{28}$$

We can compute the return from holding one unit of risky assets between t and t+1 as in Lucas' model:

$$\mathbb{E}_t\left\{1+r^k_{it+1}\right\}=\mathbb{E}_t\left\{\frac{d_{it+1}+p_{it+1}}{p_{it}}\right\}$$

2.2 Firms

In each period, the representative firm $i \in [0,n]$ has to decide how much labor to hire and how much to invest. Managers maximize the value of the firm to its owners (the representative agent) which is equal to the present discounted value of all current and future expected cash flows:

$$\max_{\{n_{it},k_{it},i_{it}\}} \mathbb{E}_t\left\{\sum_{\tau=0}^{\infty}\beta^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left(y_{it+\tau}-h_{it+\tau}w_{t+\tau}-i_{it+\tau}\right)\right\} \tag{29a}$$

$$s.t.\quad k_{it}=\left(1-S\left(i_{it}/i_{it-1}\right)\right)i_{it}+\left(1-\delta\right)k_{it-1} \tag{29b}$$

where $\beta^\tau\lambda_{t+\tau}/\lambda_t$ the marginal rate of substitution of the owners, y_{it} is the firm specific output, i_{it} is the investment in capital stock, h_{it} is the quantity of labor/hours worked input and w_t is the wage rate. The firm's capital stock k_{it} obeys an intertemporal accumulation equation with adjustment costs: $S\left(i_{it}/i_{it-1}\right)$ with $S(.)$ a positive, concave function as in Christiano et al. (2005). Concavity of the function, $S(.)$, captures the idea that changing the capital stock rapidly is more costly than changing it slowly. This specification also allows the shadow price of installed capital q_t to diverge from the price of an additional unit of capital, *i.e.*, it permits variation in Tobin's q. This cost function reads as: $S\left(i_{it}/i_{it-1}\right)=0.5\chi(i_{it}/i_{it-1}-g)^2$ with $\chi>0$ the adjustment cost parameter and is characterized by $S\left(g\right)=S'\left(g\right)=0$ and $S''\left(g\right)>0$. In addition, $\delta\in[0,1]$ is the depreciation rate of physical capital (when $\delta\rightarrow 1$, capital is as perishable as consumption goods and are perfect substitutes).

The firm does not issue new shares and finances its capital stock solely through retained earnings. The dividends to shareholders are then equal to:

$$d_{it}=y_{it}-A_t h_{it} w_t-i_{it} \tag{30}$$

Technology is given by a Cobb-Douglas production function:

$$y_{it}=z_t k^\alpha_{it-1}\left(A_t h_{it}\right)^{1-\alpha} \tag{31}$$

where $\alpha\in(0,1]$ is the share of capital k_{it-1} in the production of goods (lagged of one because of time-to-build assumption), n_{it} is the amount of hours worked involved in the production process of goods and z_t is the exogenous TFP shock, while A_t is the deterministic trend in labor augmenting technical change:

$$\begin{aligned}\log(z_t)=\rho\log(z_{t-1})+\sigma e_t,\ \text{ with } e_t\sim\mathcal{N}(0,1).\\ A_t=gA_{t-1}\text{ with } g\geq 1\end{aligned}$$

where g is the long run growth rate of the economy cause by the labor augmenting productivity trend.

The Lagrangian problem reads as follows:

$$\begin{aligned}\mathcal{L}=\sum_{\tau=0}^{\infty}\beta^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\{&\left(z_{t+\tau}k^\alpha_{it-1+\tau}\left(A_{t+\tau}h_{it+\tau}\right)^{1-\alpha}-h_{it+\tau}w_{t+\tau}-i_{it+\tau}\right)\\&+q_{t+\tau}\left[\left(1-S\left(i_{it+\tau}/i_{it-1+\tau}\right)\right)i_{it+\tau}+\left(1-\delta\right)k_{it-1+\tau}-\ k_{it+\tau}\right]\}\end{aligned}$$

where q_t is the Lagrangian multiplier on the accumulation process of physical capital. When $q_t > 1$, households are more willing to invest in capital goods rather than consuming which in turn drives up asset prices.

First order conditions are given by:

$$n_{it} : (1-\alpha) \frac{y_{it}}{h_{it}} = w_t, \tag{32}$$

$$i_{it} : q_t = 1 + q_t \frac{\partial S(.) i_{it}}{\partial i_{it}} + \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} \frac{\partial S(.) i_{it+1}}{\partial i_{it}}, \tag{33}$$

$$k_{it} : q_t = \mathbb{E}_t \left\{ \beta \frac{\lambda_{t+1}}{\lambda_t} \left(\alpha \frac{y_{it+1}}{k_{it}} + (1-\delta) q_{t+1} \right) \right\} \tag{34}$$

2.3 Equilibrium and detrending

The model is solved using assumptions similar to Lucas' model. First, the net supply of asset is zero as the country has no possibility to borrow abroad (*i.e.* the closed economy setup implies a zero current account):

$$b_t = 0.$$

In addition, the number of firms operating in the economy is normalized to one $n = 1$ as in Lucas asset pricing model:

$$p_t = \frac{1}{n} \sum\nolimits_{i=1}^{n} p_{it} \text{ and } z_t = \sum\nolimits_{i=1}^{n} z_{it}.$$

Finally, labor supply is assumed to be inelastic. However firms are allowed to adjust the real wage along the cycles via the marginal product of labor, we normalize to one the number of workers:

$$h_t = \sum\nolimits_{i=1}^{n} h_{it} = 1,$$

under this assumption, we allow the labor demand to increase constantly with labor productivity.

The model is not stationary because of the random walk component in the labor productivity A_t. To make the model stationary, we employ a standard detrending method by rewriting each quantity as:[8]

$$\tilde{c}_t = c_t/A_t, \quad \tilde{y}_t = Y_t/A_t, \quad \tilde{k}_t = k_t/A_t, \quad \tilde{\imath}_t = i_t/A_t, \quad \tilde{\lambda}_t = \lambda_t/A_t^{-\gamma}.$$

The trick is to remove all the A_t of our model and rewrite it in terms of variations, denoted $\Delta A_t = g$ as g is constant.

The marginal utility of consumption can be rewritten by:

$$\lambda_t \frac{A_t^{-\gamma}}{A_t^{-\gamma}} = (c_t - hc_{t-1})^{-\gamma} - \beta h (\mathbb{E}_t \{c_{t+1}\} - hc_t)^{-\gamma}$$

Since $\lambda_t / A_t^{-\gamma}$, then:

$$\tilde{\lambda}_t = \frac{(c_t - hc_{t-1})^{-\gamma}}{A_t^{-\gamma}} - \beta h \frac{(\mathbb{E}_t \{c_{t+1}\} - hc_t)^{-\gamma}}{A_t^{-\gamma}}$$

[8] Prices are not affected in the steady state by productivity.

which can be rewritten as:

$$\tilde{\lambda}_t = \left(\tilde{c}_t - \frac{h}{g}\tilde{c}_{t-1}\right)^{-\gamma} - \beta h \left(g\mathbb{E}_t\left\{\tilde{c}_{t+1}\right\} - h\tilde{c}_t\right)^{-\gamma}$$

In addition, the pricing kernel reads as:

$$M_{t,t+1} = \tilde{\beta}\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}$$

within $\tilde{\beta} = \beta g^{-\gamma}$.

Detrending is straightforward for other equations:[9]

$$\tilde{k}_t = \left(1 - S\left(g\tilde{\imath}_t/\tilde{\imath}_{t-1}\right)\right)\tilde{\imath}_t + \frac{(1-\delta)}{g}\tilde{k}_{t-1}, \tag{35}$$

$$\tilde{w}_t = (1-\alpha)\,\tilde{y}_t, \tag{36}$$

$$q_t = 1 + q_t\frac{\partial S(.)\,\tilde{\imath}_t}{\partial i_t} + \tilde{\beta}\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}q_{t+1}\frac{\partial S(.)\,\tilde{\imath}_{t+1}}{\partial i_t}, \tag{37}$$

$$q_t = \mathbb{E}_t\left\{\tilde{\beta}\frac{\tilde{\lambda}_{t+1}}{\tilde{\lambda}_t}\left(\alpha g\frac{y_{t+1}}{k_t} + (1-\delta)\,q_{t+1}\right)\right\}, \tag{38}$$

$$\tilde{y}_t = \tilde{c}_t + \tilde{\imath}_t, \tag{39}$$

$$\tilde{d}_t = \tilde{y}_t - w_t - \tilde{\imath}_t, \tag{40}$$

$$\tilde{y}_t = z_t g^{-\alpha}\tilde{k}^{\alpha}_{it-1}. \tag{41}$$

2.4 Calibration

We slightly use a different calibration compared to Lucas' model. In particular, to generate a standard deviation of consumption growth close to the empirical one, we set $\rho = 0.8$ and $\sigma = 0.17$. The discount factor $\beta = 0.99$ and the trend $\gamma = 1.02$ are left unchanged. The depreciation rate of physical capital is set at δ =2%, the capital share in production to $\alpha = 0.36$, the adjustment cost of investment to $\chi = 0.5$ and the habits to $h = 0.8$. These parameters are in line with Jermann (1998).

Exercise 4

1. Fill the column 'near-Jerman' in Table 1 (page 17) using the nj.mod by simulating the model using a second order approximation to the policy function.
2. Investigate which paradox is solved.
3. Highlight why this model has clear failures, in particular regarding the riskless interest rate.
4. Does the adjustment cost χ amplify the equity premium? Provide some insight regarding the role χ on the equity premium.

[9] Note that $\partial S(.)\,\tilde{\imath}_t/\partial i_t = 0.5\chi g^2\left[1 + 3\left(\Delta\tilde{\imath}_t - 4\right)\Delta\tilde{\imath}_t\right]$ and $\partial S(.)\,\tilde{\imath}_{t+1}/\partial i_t = -\chi g^3\left[\Delta\tilde{\imath}_t - 1\right]\Delta\tilde{\imath}_t^2$.

3. Recursive utility in a pure exchange economy

In this section, I show a way to fix the 'risk-free rate puzzle' of Weil (1989) by using time-separable utility functions. Here, I extend the model of Lucas Jr (1978) to account for time-preference utility functions as defined by Epstein and Zin (1989).

3.1 A discussion on recursive utility

One of the issues that seem to be crucial in the puzzle is that the CES utility function mixes the time structure of preferences and the aversion for risk. Both are measured by (functions of) the same parameter. Recall that in Lucas Jr (1978), the welfare index reads as: $\mathcal{W}_t = \mathbb{E}_t \sum_{\tau=0}^{\infty} \beta^\tau c_{t+\tau}^{1-\gamma}/(1-\gamma)$, which can be rewritten recursively as:

$$\mathcal{W}_t = \frac{c_t^{1-\gamma}}{(1-\gamma)} + \beta \mathbb{E}_t \mathcal{W}_{t+1}$$

Under these preferences, we have seen that up to second order, the equity premium and the risk-free rate were determined by:

$$E\left[\ln EP_t\right] = \gamma cov(\ln \Delta\tilde{c}_{t+1}, r_{t+1}^k),$$
$$E\left[r_t\right] = -\ln\beta + \gamma\ln(g) - 0.5\gamma^2 var\left(\Delta\tilde{c}_{t+1}\right)$$

To replicate the observed equity premium $E\left[\ln EP_t\right]$, we need a high elasticity of intertemporal substitution γ (equity premium puzzle) while to replicate the risk-free rate $E\left[r_t\right]$, we need a low coefficient of risk aversion γ (risk-free rate puzzle). The solution of this problem is straightforward, we have to separate the coefficient of relative risk aversion γ from the elasticity of intertemporal substitution (denoted now ψ instead of γ).

The Epstein and Zin (1989) preferences reads as a CES function:

$$\mathcal{W}_t = \left[(1-\beta)\, c_t^{1-\psi} + \beta\mathbb{E}_t \left\{\mathcal{W}_{t+1}^{1-\gamma}\right\}^{\frac{1-\psi}{1-\gamma}}\right]^{\frac{1}{1-\psi}}. \tag{42}$$

This CES function allows the welfare index $\mathcal{W}_t$ to be a combination of current utility $c_t^{1-\psi}$ and expected welfare/utility $\mathbb{E}_t\{\mathcal{W}_{t+1}^{1-\gamma}\}^{\frac{1-\psi}{1-\gamma}}$, with the discount factor 1-β that represents the share of current utility in the welfare index of the household and β the share of future utility. One may notice that $\beta \to 1$, households are more concerned by future utility than the current one. The CES allows to generate complementarity/substitutability between current and future utility, and is governed by the value of ψ and γ:

- if $\psi = \gamma$, the welfare boils down to:

  $$\mathcal{W}_t^{1-\psi} = c_t^{1-\psi} + \beta\mathbb{E}_t\mathcal{W}_{t+1}^{1-\psi},$$

  the recursion becomes linear as in the standard time-separable power utility model we had in Lucas Jr (1978)'s model.

- if $\psi > \gamma$, agents are more concerned about uncertainty than they are about predictable variation in consumption. This leads to an amplification of the impact of consumption fluctuations on the stochastic discount factor, as households seek some insurance against expected consumption decline (in recessions for example).

3.2 An application to Lucas' model

Considering Epstein and Zin (1989) preferences, the welfare index reads as follows:

$$\mathcal{W}_t = \left[\left(1-\tilde{\beta}\right) c_t^{1-\psi} + \beta \mathbb{E}_t \left\{ \mathcal{W}_{t+1}^{1-\gamma} \right\}^{\frac{1-\psi}{1-\gamma}} \right]^{\frac{1}{1-\psi}} \tag{43}$$

where β is the discount factor, and $\tilde{\beta}$ is a scaling constant. We allow long run growth in the model thus requiring to choose $\tilde{\beta} = \beta g^{-\gamma}$ to obtained a balanced (detrended) steady state.

The budget constraint is left unchanged:

$$c_t + b_t + \textstyle\sum_{i=1}^{n} p_{it} z_{it} = (1 + r_{t-1}) b_{t-1} + \sum_{i=1}^{n} \left[d_{it} + p_{it} \right] z_{it-1} \tag{44}$$

We cannot compute first order conditions directly through a Lagrangian problem because of the distortion incurred by Zin Epstein's preferences in the objective function of the optimization problem. However, it is possible to solve the optimization problem using a Bellman equation as defined in Equation 43.

To obtain first-order conditions from preferences, we must calculate:

$$\begin{aligned}
\frac{\partial \mathcal{W}_t}{\partial c_t} &= \left(1-\tilde{\beta}\right) \mathcal{W}_t^{\psi} c_t^{-\psi}, \\
\frac{\partial \mathcal{W}_t}{\partial c_{t+1}} &= \left(1-\tilde{\beta}\right) \beta \mathcal{W}_t^{\psi} \left(\frac{\mathcal{W}_{t+1}}{\mathbb{E}_t \left\{ \mathcal{W}_{t+1}^{1-\gamma} \right\}^{\frac{1}{1-\gamma}}} \right)^{\psi-\gamma} c_{t+1}^{-\psi},
\end{aligned}$$

This allows us to get the stochastic discount factor:

$$\begin{aligned}
M_{t,t+1} &= \frac{\partial \mathcal{W}_t / \partial c_{t+1}}{\partial \mathcal{W}_t / \partial c_t} \\
&= \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\psi} \left(\frac{\mathcal{W}_{t+1}}{\mathcal{R}_t} \right)^{\psi-\gamma}
\end{aligned}$$

where $\mathcal{R}_t = \mathbb{E}_t \{ \mathcal{W}_{t+1}^{1-\gamma} \}^{\frac{1}{1-\gamma}}$.

It is important to stress that when $\psi = \gamma$, we can directly observe that $M_{t,t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\psi}$ as in the Lucas' model, while when $\psi > \gamma$, risk matters as expected utility $\mathcal{W}_{t+1}$ affects negatively the stochastic discount factor by amplifying consumption cycles.

Applying the standard detrending method with $\tilde{\mathcal{W}}_t = \mathcal{W}_t / A_t$ and $\tilde{\mathcal{R}}_t = \mathcal{R}_t / A_t$, our Lucas model includes now two new detrended equations:

$$\begin{aligned}
\tilde{\mathcal{W}}_t &= \left[\left(1-\tilde{\beta}\right) \tilde{c}_t^{1-\psi} + \beta \tilde{\mathcal{R}}_t^{1-\psi} \right]^{\frac{1}{1-\psi}}, \\
\tilde{\mathcal{R}}_t &= \mathbb{E}_t \left\{ (g \mathcal{W}_{t+1})^{1-\gamma} \right\}^{\frac{1}{1-\gamma}},
\end{aligned}$$

while we modify the pricing kernel:

$$M_{t,t+1} = \beta g^{-\gamma} \left(\frac{\tilde{c}_{t+1}}{\tilde{c}_t} \right)^{-\psi} \left(\frac{\tilde{\mathcal{W}}_{t+1}}{\tilde{\mathcal{R}}_t} \right)^{\psi-\gamma}$$

Under this setup, it is important to notice that with:

$$\mathbb{E}_t \{M_{t,t+1}\} (1 + r_t) = 1,$$

Applying logs (without making any approximation or computing the mean):

$$r_t = -\ln\beta + \gamma\ln(g) + \psi(\Delta\ln\tilde{c}_{t+1}) - (\psi - \gamma)\ln\left(\frac{\tilde{\mathcal{W}}_{t+1}}{\tilde{\mathcal{R}}_t}\right)$$

It's easy to see how Zin Epstein preferences fixed the riskless paradox, as it can combine low value of risk aversion γ associated to the long run growth of consumption $\ln(g)$, and high value of the marginal rate of intertemporal substitution ψ modelling precautionary saving regarding expected consumption growth $\Delta\ln\tilde{c}_{t+1}$. Finally the last term $\ln(\tilde{\mathcal{W}}_{t+1}/\tilde{\mathcal{R}}_t)$ requires a third order approximation to the model policy function to affect the riskless-rate and can be interpreted as the expected uncertainty about future shocks. In particular if $\psi > \gamma$ (and up to 3rd order), the expected realization of shocks affects negatively the riskless rate by forcing households to bring more liquidities on financial market to shield against future uncertainty and smooth more consumption.

Regarding the calibration, we simply assume that $\gamma = 0.5$ and $\psi = 5$ (note that $\psi > \gamma$) while all other parameters are left unchanged.

Exercise 5

1. Fill the last column column 'recursive utility' in Table 1 using the recursive utility model `lucasRU.mod`.

2. Which model better replicate the data (or is more convincing) now that Table 1 is completely filled?

3. Does a third order approximation has important implications on the equity premium and the riskless rate?

4. Discuss the situation when $\psi < \gamma$ using simulations of the model.

	Data	Lucas simple	near-Jerman	Recursive Utility
Standard deviations				
$\sigma(c_t)$ - consumption	....	....	....	....
$\sigma\left(r_t^k\right)$ - risky rate	....	....	....	....
$\sigma(r_t)$ - riskless rate	....	....	....	....
$\sigma(EP_t)$ - equity premium	....	....	....	....
Means				
$E(c_t)$ - consumption	....	....	....	....
$E\left(r_t^k\right)$ - risky rate	....	....	....	....
$E(r_t)$ - riskless rate	....	....	....	....
$E(EP_t)$ - equity premium	....	....	....	....
Ratios				
$\frac{E(EP_t)}{\sigma\left(r_t^k\right)}$ - sharpe ratio	....	....	....	....

Table 1: Exercise 2, 3, 4 business cycle statistics (to fill)

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